JOURNAL OF ALGEBRA 97, 116–129 (1985)

Affine Noetherian PI-Rings Have Enough Clans

BRUNO J. MÜLLER

McMaster University, Hamilton, Ontario L8S 4K1, Canada

Communicated by A. W. Goldie

Received July 2, 1984

A noetherian PI-ring which is affine over the centre, can be localized at every (finite or infinite) link connectivity component $\mathscr{P}$, in the sense that the multiplicative set $\mathscr{C}(\mathscr{P})$ satisfies the Ore condition and determines $\mathscr{P}$.

1. INTRODUCTION

In [13] we obtained the first result pertaining to the localization of a noetherian ring at an infinite set of prime ideals, namely that an affine noetherian PI-algebra over an uncountable field can be localized at every link connectivity component (i.e., clique). In the meantime, this theorem has been extended considerably, first by Brown who proved an anlogue for enveloping algebras of solvable Lie algebras [2], then by Jategaonkar who established it for arbitrary noetherian PI-algebras and for noetherian algebras with the second layer condition and completely prime ideals [6, 8.45 and 8.40], and finally by Warfield who dealt with noetherian algebras with the second layer condition and a bound on the Goldie rank of the prime ideals in any particular clique [20]; cf. also the recent surveys [3] and [18]. But all these generalizations require our uncountability hypothesis, and we are not aware of any prior result that does without it.

The removal of this uncountability hypothesis is the object of the present paper. The proof is fairly technical. An easy reduction allows us to confine ourselves again to algebras over a field. The main step consists of showing that all the prime ideals of a clique can be described, in a technical sense, over a relatively small subfield. This works because of the affineness assumption, and because all links in a noetherian PI-ring can be derived from a finite collection of ideals, as was already observed in [11]. (There is a superficial resemblance with Stafford's procedure (cf. [3, 4.2]), but his situation allows the shortcut of choosing the subfield directly large enough to include data describing all the prime ideals of the clique, without worrying about links.) Once this lemma is established, the theorem follows

0021-8693/85 $3.00

immediately if the field is algebraically closed, and one can show in addition that the quotient ring has stable range one. Over a general field, we have to add a symmetrization procedure which relies on central polynomials, and yields no information about the stable range, at this point.

In view of applications, we establish the theorem for finite unions of cliques. In contrast to the case of finite cliques, we know of no formal procedure that would yield the localizability of such finite unions from the knowledge of the localizability of the single cliques.

Whether the affineness assumption can be circumvented, remains open.

2. Terminology

We consider (left- and right-) noetherian rings. Module means right-module, but ideal means two-sided ideal, unless specified otherwise. $Z(R)$ is the centre of the ring R. The annihilators of a left-module N and a right-module M in the ring R are denoted by $l(N) = l_R(N)$ and $r(M) = r_R(M)$.

An affine algebra over a field k is a finitely generated k-algebra. An affine ring is a finitely generated algebra over the centre.

For a field F, the algebraic closure is denoted by $\tilde{F}$, and the dimension of an F-vectorspace V by $[V:F]$. The transcendence degree over a subfield k is $\text{t.d.}_k F$.

Reference [6] can serve as a comprehensive reference to localization.

3. Generalities

(1) Two types of links between the prime ideals of a noetherian ring R have been studied: An (*ideal*) *link* $Q \sim P$ is given by ideals $A \subsetneq B$ such that B/A is a torsionfree R/Q-left- and R/P-right-module. A *second layer link* $Q \rightsquigarrow P$ is the special case where $QP \subset A \subsetneq B = Q \cap P$ [6]. The corresponding connectivity components on spec R are called *cliques* and *second layer cliques*. A set of prime ideals which is closed under (second layer) links is called (*second layer*) *stable*.

Ideal links can be traced to [5]. For the noetherian *PI*-rings which interest us here, and more generally for *FBN*-rings, second layer links appear under the name of "short left links", and ideal links as "long left links" and "bimodule links" in [11–14]. According to Proposition 4 of [11] and Corollary 16 of [12], cliques and second layer cliques coincide for these rings.

(2) For a set $\mathscr{P}$ of prime ideals, we let $\mathscr{C}(\mathscr{P})$ denote the multiplicative subset $\bigcap\{\mathscr{C}(P) \colon P \in \mathscr{P}\}$ of R. Associated with $\mathscr{P}$ are two, comparable, tor-

siontheories for right- (and analoguously for left-) modules, namely, $\tau_{\mathscr{C}(\mathscr{P})} \leqslant \bigcap_{P \in \mathscr{P}} \tau_{\mathscr{C}(P)}$. The requirement of their equality is the important *intersection condition*, which was first used in [13, Sect. 6]. In technical terms, the right-intersection condition asks that for any right-deal I, $r^{-1}I \cap \mathscr{C}(P) \neq \varnothing$ for all $P \in \mathscr{P}$ and $r \in R$ yields $I \cap \mathscr{C}(\mathscr{P}) \neq \varnothing$. One can usually deduce this conclusion from the weaker assumption $I \cap \mathscr{C}(P) \neq \varnothing$ for all $P \in \mathscr{P}$.

(3) One is interested in localizations of a noetherian ring R at Ore sets S. Such an S determines the set $\mathscr{P} = \{P \in \operatorname{spec} R\colon R \cap S = \varnothing\}$, which is hereditary (i.e., closed under smaller prime ideals), stable, and corresponds to the spectrum of the quotient ring R_S [6, 5.4].

For an *FBN*-ring, we called a set $\mathscr{P}$ of prime ideals *classical* if it arises in this way [13, Sect. 6]. We showed that a set $\mathscr{P}$ is classical if and only if it is hereditary, stable, and satisfies the intersection condition, and that then $\mathscr{C}(\mathscr{P}) = \mathscr{C}(\max \mathscr{P})$ is the largest Ore set leading to the same quotient ring. (The assumption "with locally finite links" in [13, Corollary 17] is redundant.) Similar ideas appear, independently, in [19].

Thus, establishing our theorem that every clique of an affine noetherian *PI*-ring is classical, amounts to verifying the intersection condition.

(4) For an arbitrary noetherian ring, Jategaonkar calls a set $\mathscr{X}$ *classical* if $\mathscr{C}(\mathscr{X})$ satisfies the Ore condition and the quotient ring has these three pleasant properties: the prime ideals corresponding to the members of $\mathscr{X}$ have artinian factor rings; there are no other primitive ideals; the injective hulls of the simple modules are the unions of their socle sequences. He proves that all of this holds if and only if $\mathscr{X}$ is stable and satisfies the second layer, incomparability and intersection conditions [6, 8.35].

It is not difficult to verify that a hereditary set $\mathscr{P}$ (or rather the corresponding $\mathscr{X} = \max \mathscr{P}$) is classical if and only if it arises from an Ore set whose quotient ring has the three pleasant properties. (For an *FBN*-ring, these properties hold automatically, and thus the terminology is consistent.)

As remarked in [6, after 7.31], a clique with the second layer condition is second layer link connected. On the other hand, a careful reading of the proof of [6, 8.35] reveals that a second layer clique with second layer and intersection conditions is stable. It follows that, in the presence of the second layer and intersection conditions, cliques and second layer cliques coincide in general.

In view of these observations, and the established use of the term for finite cliques (cf. [10, 11] and [6, 8.9]), we propose to call a clique with the second layer and intersection conditions a *clan* (i.e., a *minimal classical* set).

In this terminology we are about to establish that every clique of an affine noetherian PI-ring is a clan, to say that such a ring has "enough clans."

4. The Fields $K^{(N)}$

In this section we prepare some elementary field theoretical material for later use.

We consider a field K which is purely transcendental over its prime field $\mathbb{P}$, and with algebraic closure $\tilde{K}$. For any natural number N we define $K^{(N)}$ to be the subset of $\tilde{K}$ consisting of those elements a for which there exists a finite tower $K = F_0 \subset F_1 \subset \cdots \subset F_s \subset \tilde{K}$ of intermediate fields with $a \in F_s$ and $[F_i : F_{i-1}] \leqslant N$ for all $i = 1,..., s$.

LEMMA 1. (1) *$K^{(N)}$ is a field*; (2) *$K^{(N)}$ is closed under extensions of degree $\leqslant N$*; (3) *the various $K^{(N)}$ form an increasing sequence with union $\tilde{K}$*; (4) *for every element $\eta \in \tilde{\mathbb{P}}$ of prime degree $> N$ over $\mathbb{P}$ and for every non-zero polynomial g of degree $\leqslant N$ over $K^{(N)}$ one has $g(\eta) \neq 0$.*

Proof. It is easy to see that for any finite collection of elements $a_0,..., a_n \in K^{(N)}$ there exists a tower $K = F_0 \subset \cdots \subset F_s \subset \tilde{K}$ with $[F_i : F_{i-1}] \leqslant N$ and $a_0,..., a_n \in F_s$. Then (1) follows immediately. So does (2), by considering the coefficients of the minimal polynomial of an element of degree $\leqslant N$ over $K^{(N)}$. Statement (3) is trivial.

As to (4), suppose one has $g(\eta) = 0$ in the situation described. Certainly η has degree $\leqslant N$ over $K^{(N)}$, and hence belongs to $K^{(N)}$ by (2). Thus one has $K = F_0 \subset F_1 \subset \cdots \subset F_s \subset \tilde{K}$ with $\eta \in F_s$ and $[F_i : F_{i-1}] \leqslant N$, and therefore the degree of η over K divides $\prod_{i=1}^{s} [F_i : F_{i-1}]$. As $K/\mathbb{P}$ is purely transcendental, one derives readily that η has the same degree over K and $\mathbb{P}$. Since the latter is a prime number $> N$, one obtains a contradiction. ∎

5. Absolutely Irreducible Representations

We proceed to collect information about these representations. Most of the material here can be found in [15].

An *absolutely irreducible representation* (AI-representation) of a ring R (of degree n, over a field F, for an ideal P) is a ringhomomorphism $\varepsilon: R \to M_n(F)$, into the ring of $n \times n$-matrices, such that $\ker \varepsilon = P$ and $(\operatorname{im} \varepsilon)F = M_n(F)$.

The following facts are well known [15, 55–57]:

(I) An AI-representation for an ideal P exists if and only if P is a prime ideal of finite PI-degree, and then the degree of the representation equals the PI-degree of P.

(II) An AI-representation over a field F induces one over every extension field.

(III) If $\varepsilon: R \to M_n(F)$ is an AI-representation for P, and if $c \in \mathscr{C}(P)$, then c is regular (and hence invertible) in $M_n(F)$. Consequently the quotient ring of R/P embeds into $M_n(F)$.

(IV) Given any two AI-representations $\varepsilon_i: R \to M_n(F_i)$ for the same P, and embeddings of F_1 and F_2 into a third field F which are compatible with the embeddings of $Z(R/P)$ into the F_i induced by the ε_i, then there exists an automorphism σ of $M_n(F)$ such that $\sigma\varepsilon_1 = \varepsilon_2$.

If R is an algebra over a field k, then F is an extension of k, and ε is a k-algebra homomorphism, in a natural way. Then these additional facts are true:

(V) For any AI-representation $\varepsilon: R \to M_n(F)$ for P, one has $\text{t.d.}_k F \geqslant \text{t.d.}_k Z(R/P)$; and there exists a representation for which equality holds.

(VI) If R is affine over k and if P is a prime ideal of finite PI-degree, then the classical Krull dimension of R/P, the transcendence degree $\text{t.d.}_k Z(R/P)$, and the Gelfand–Kirillov dimension $GK_k(R/P)$ all coincide [8] and are finite. (We shall denote this quantity by $\dim P$.)

LEMMA 2. *Let $R = k\{S\}$, S finite, be an affine k-algebra, let k^* be an algebraic subfield of k, and $R^* = k^*\{S\}$. Let P be a prime ideal of R of finite PI-degree, and $P^* = P \cap R^*$. Then P^* is a prime ideal of R^*, and* $\dim P^* = \dim P$.

Proof. Due to $R = R^*k$, P^* is a prime ideal of R^*. For the same reason, the map $R^*/P^* \cong R^* + P/P \subset R/P$ embeds $Z(R^*/P^*)$ into $Z(R/P)$. Consequently, and since k/k^* is algebraic, we have $\dim P^* = \text{t.d.}_{k^*} Z(R^*/P^*) \leqslant \text{t.d.}_{k^*} Z(R/P) = \text{t.d.}_k Z(R/P) = \dim P$.

The converse inequality, $\dim P = GK_k(R/P) \leqslant GK_{k^*}(R^*/P^*) = \dim P^*$, follows immediately from the definition the Gelfand–Kirillov dimension, $GK_k(k\{S\}) = \limsup \log[S^n k : k]/\log n$, upon observing $[S^n k : k] \leqslant [S^n k^* : k^*]$. ∎

LEMMA 3. *With $K^{(N)}$ as in Section 4, any $K^{(N)}$-algebra A with $[A : K^{(N)}] \leqslant N$ possesses absolutely irreducible representations over $K^{(N)}$ for all maximal ideals.*

Proof. As A is artinian, $A/M \cong M_m(D)$ holds, with a division ring D, for every maximal ideal M. Clearly $[D : K^{(N)}] \leqslant [A : K^{(N)}] \leqslant N$. Therefore any $d \in D$ has degree $\leqslant N$ over $K^{(N)}$. As $K^{(N)}$ is closed under such extensions, by Lemma 1, we conclude $D = K^{(N)}$. Thus $A \to A/M \cong M_m(K^{(N)})$ is the desired AI-representation. ∎

6. Remarks about Links

We recall that links in a noetherian PI-ring preserve Krull dimension. The first lemma is essentially contained in [11, Theorem 7 and subsequent discussion], but we give a short proof.

LEMMA 4. *Let R be a noetherian PI-ring of PI-degree d. Let $N_i = \bigcap\{P : PI\text{-degree}(P) \leqslant i\}$, put $B = N_1 \oplus \cdots \oplus N_d$, and let $\bar{B}$ denote the right-R/P-torsion radical factor module of B/BP. Then, for every second layer link $Q \rightsquigarrow P$ between distinct prime ideals, $Q \supset l(\bar{B})$ holds.*

Proof. Let $j = \max\{PI\text{-degree}(Q), PI\text{-degree}(P)\}$, and $N = N_j$. Let the second layer link $Q \rightsquigarrow P$ be given by $QP \subset A \subsetneqq Q \cap P$.

If $N \subset A$, then the second layer link is preserved in the factor ring R/N. But R/N is semiprime, and one of Q/N, P/N is of maximal PI-degree, hence localizable [17; 15, VII.1.6(1)], hence not linked to any other prime ideal. This contradiction shows that $N \not\subset A$ holds.

We conclude $N + A/A \neq 0$ and therefore $Q = l(N + A/A)$. As $N + A/A \subset Q \cap P/A$ is right-R/P-torsionfree, we obtain a natural epimorphism $\bar{B} \to N + A/A$, and we conclude $l(\bar{B}) \subset l(N + A/A) = Q$. ∎

Conversely one can show that, if $\dim Q = \dim P$ and $Q \supset l(\bar{B})$ hold, then there is an ideal link $Q \sim P$. The next lemma will be used in Section 8.

LEMMA 5. *Second layer links between distinct prime ideals are not affected by Ore localization and by passage to a factor ring modulo a centrally generated ideal.*

Proof. It is well known, and easy to see, that no links are created nor destroyed by an Ore localization. Trivially, no links are created by passage to any factor ring.

Consider now a second layer link $Q \rightsquigarrow P$ with $Q \neq P$ given by $QP \subset A \subsetneqq Q \cap P$, and a centrally generated ideal I contained in P. Note that I commutes with any other ideal. If $I \subset A$, then $I \subset Q$, and $\overline{QP} \subset \bar{A} \subsetneqq \bar{Q} \cap \bar{P}$ provides a second layer link $\bar{Q} \rightsquigarrow \bar{P}$ in $\bar{R} = R/I$. If $I \not\subset A$, then $I(Q \cap P) \subset IQ = QI \subset QP \subset A$, hence $I \subset l(Q \cap P/A) = Q$. Thus $0 \neq I + A/A \subset Q \cap P/A$ and therefore $Q = l(I + A/A) = r(I + A/A) = P$, a contradiction. ∎

Remark. Examples show that second layer self links may disappear under passage to a factor ring modulo a centrally generated ideal. For instance, let

$$A = \begin{pmatrix} k[x] & k[x] \\ xk[x] & k[x] \end{pmatrix}, \qquad P = \begin{pmatrix} xk[x] & k[x] \\ xk[x] & k[x] \end{pmatrix}$$

and $\bar{A} = A/P$. Then the prime ideal $P \rtimes \bar{A}$ of the split extension $R = A \rtimes \bar{A}$ has a self link, which disappears in the factor ring modulo the ideal generated by the central element $(0, \bar{1})$.

7. The Main Lemma

Suppose we are given a field k and a natural number t. Denote the prime field of k by $\mathbb{P}$, and let T be a transcendence base for k over $\mathbb{P}$. Pick independent transcendentals $z_1, ..., z_t$ over k and put $K = \mathbb{P}(T, z_1, ..., z_t)$. Then we have

$$\begin{array}{ccccc} \mathbb{P} \subset & \mathbb{P}(T) & \subset & K \\ & \cap & & \cap \\ & k & \subset & \tilde{K}, \end{array}$$

where the vertical inclusions are algebraic, and K is purely transcendental over $\mathbb{P}$. Let $K^{(N)}$ be constructed from K as in Section 4. With this notation, the fundamental technical result is the following:

Lemma 6. *Let R be an affine noetherian PI-algebra over a field k, and let $\mathscr{P}$ be a clique of Krull dimension t. For every finite subset S of R there exist absolutely irreducible representations $\varepsilon_P \colon R \to M_{n_P}(\tilde{K})$ such hat $\varepsilon_P(s) \in M_{n_P}(K^{(N)})$ holds, for all $s \in S$, $P \in \mathscr{P}$ and all sufficiently large natural numbers N.*

Proof. Enlarging S if necessary, we assume that it generates R as k-algebra: $R = k\{S\}$.

Recall that d stands for the PI-degree of R (i.e., the maximum of the PI-degrees of the prime ideals of R). Recall also the bimodule $B = N_1 \oplus \cdots \oplus N_d$ used in Lemma 4, and note that B is a k-algebra bimodule and is finitely generated on both sides. Thus we can fix elements $b_1, ..., b_g \in B$ such that $B = \sum_{i=1}^{g} Rb_i = \sum_{i=1}^{g} b_i R$. For each $s \in S$ we obtain $sb_i = \sum_{j=1}^{g} b_j r_{jis}$ and $b_i s = \sum_{j=1}^{g} r'_{jis} b_j$. The finitely many elements $r_{jis}, r'_{jis} \in R = k\{S\}$ can be written as polynomials in the members of S with coefficients in k. (This is the instance where the affineness of R is used most crucially.)

Next we fix a particular $P_0 \in \mathscr{P}$, and an AI-representation $\varepsilon_0 \colon R \to M_{n_0}(\tilde{K})$ for P_0 (which exists by (V) of Sect. 5).

From Lemma 1 it is clear that, for sufficiently large N, the following are true:

(N1) $r_{jis}, r'_{jis} \in (k \cap K^{(N)})\{S\}$;

(N2) $\varepsilon_0(s) \in M_{n_0}(K^{(N)})$ for all $s \in S$;

(N3) $N \geqslant (gd^2)^2$.

We fix such an N, and abbreviate $K^{(N)}=K^*$, $K^{(N)}\cap k=k^*$, $k^*\{S\}=R^*$. Then k/k^* is algebraic, R^* is an affine PI-algebra over k^* of PI-degree $\leqslant d$ (but not necessarily noetherian), and $R^*\subset R$ and $R^*k=R$ hold (and hence Lemma 2 applies). For any ideal I of R, I^* will denote the ideal $I\cap R^*$ of R^*. We also write $B^*=\sum_{i=1}^{g} b_iR^*$ and ${}^*B=\sum_{i=1}^{g} R^*b_i$. B^* and ${}^*B$ are k^*-algebra bimodules over R^*, and are g-generated on the right respectively left. $B^*k=B={}^*Bk$ holds. In this setting we show first:

LEMMA 7. *For every $P\in\mathscr{P}$, R^* has an absolutely irreducible representation for P^* over K^*.*

Proof of Lemma 7. By (N2), the restriction $\varepsilon_0\mid R^*$ maps into $M_{n_0}(K^*)$. As $\varepsilon(R^*)\tilde{K}=\varepsilon(R^*)k\tilde{K}=\varepsilon(R^*k)\tilde{K}=\varepsilon(R)\tilde{K}=M_{n_0}(\tilde{K})$ is true, $\varepsilon(R^*)$ contains a $\tilde{K}$-basis of $M_{n_0}(\tilde{K})$, which consists of n_0^2 elements. This basis stays linearly independent over the subfield K^*, and is therefore a K^*-basis of $M_{n_0}(K^*)$. It follows that $\varepsilon_0\mid R^*$ is an AI-representation of R^* over K^* for P_0^*.

If we use now notation and statement of Lemma 4 and its left–right-analogue, we see that Lemma 7 follows once we prove the following:

Claim 1. If Q, P are prime ideals of R with $\dim Q=\dim P=t$, if $Q\supset l(\bar{B})$ holds, and if R^* has an AI-representation over K^* for P^*, then it has one for Q^*.

Proof of Claim 1. We are given an AI-representation $\varepsilon^*\colon R^*\to M_n(K^*)$ for P^*. Let $C^*=Z(R^*/P^*)$. Then C^* embeds into K^* via ε^*. The quotient ring $R^*/P^*\otimes_{C^*} qf(C^*)$ of R^*/P^* embeds into $M_n(K^*)$ (Fact (III) of Sect. 5), and consequently $R^*/P^*\otimes_{C^*}K^*\cong M_n(K^*)$ holds.

The module $\tilde{B}=B^*/B^*P^*\otimes_{C^*}K^*$ is a well defined $R^*\otimes_{k^*}K^*-R^*/P^*\otimes_{C^*}K^*$-bimodule. As right-module it is g-generated, and we conclude $[\tilde{B}:K^*]\leqslant gn^2$. We then obtain $[\mathrm{endo}_{K^*}(\tilde{B}):K^*]\leqslant(gn^2)^2\leqslant(gd^2)^2\leqslant N$.

Let A be the natural image of $R^*\otimes_{k^*}K^*$ in $\mathrm{endo}_{K^*}(\tilde{B})$, via the left-module structure of $\tilde{B}$. This is a K^*-subalgebra with $[A:K^*]\leqslant[\mathrm{endo}_{K^*}(\tilde{B}):K^*]\leqslant N$. Lemma 3 yields AI-representations $A\to M_{m_j}(K^*)$ for all maximal ideals M_j of A.

We consider the natural k^*-algebra homomorphism $\lambda\colon R^*\to R^*\otimes_{k^*}K^*\to A$, and we set $V_j=\lambda^{-1}(M_j)$. As we have $\lambda(R^*)K^*=A$, we obtain AI-representations $R^*\to A\to M_{m_j}(K^*)$ over K^* for the V_j. In particular, the V_j are prime ideals of R^*.

We can index the maximal ideals of A (with repetitions) in such a way that $\prod M_j=0$. Then $\prod V_j\subset\ker\lambda\subset Q^*$ results from the next Claim 2. We derive the existence of j_0 such that $V_{j_0}\subset Q^*$. From Lemma 2 and the facts (V) and (VI) of Sect. 5 we deduce $t=\dim Q=\dim Q^*\leqslant\dim V_{j_0}=\mathrm{t.d.}_{k^*}Z(R^*/V_{j_0})\leqslant\mathrm{t.d.}_{k^*}K^*=t$. The resulting equality implies $V_{j_0}=Q^*$ and shows the existence of an AI-representation for Q^*. This completes the proof of Claim 1, and hence of Lemma 7, except for the verification of the following:

Claim 2. $\ker \lambda \subset Q^*$.

Proof of Claim 2. Due to $R^*k = R$, the map $R^*/P^* \cong R^* + P/P \subset R/P$ embeds C^* into $C = Z(R/P)$. The nonzero elements of C are regular modulo P. We obtain a commutative diagram of natural maps

$$\begin{array}{ccc} B^*/B^*P^* & \longrightarrow & B^*/B^*P^* \otimes_{C^*} K^* = \tilde{B} \\ \downarrow & & \downarrow \\ B/B_P & \longrightarrow & B/BP \otimes_C \tilde{K}. \end{array}$$

If $a \in \ker \lambda = l_{R^*}(B)$, then $ab \otimes 1 = 0$ holds in $\tilde{B}$ and hence in $B/BP \otimes_C \tilde{K}$, for all $b \in B^*$. Therefore the image of ab in B/BP lies in the kernel of $B/BP \to B/BP \otimes_C \tilde{K}$, which is the C-torsion submodule [4, VII.2.1]. We conclude $a\bar{b} = 0$ in $\bar{B}$. Using now $B = B^*k$ we deduce $a \in l(\bar{B}) \subset Q^*$. ∎

It is now easy to derive Lemma 6 from Lemma 7: By Fact (V) we have an AI-representation $\varepsilon_P: R \to M_{n_P}(\tilde{K})$ for every $P \in \mathscr{P}$. Then $\varepsilon_P \mid R^*$ is an AI-representation for P^* over $\tilde{K}$. On the other hand, Lemma 7 yields an AI-representation $\varepsilon_P^*: R^* \to M_{n_P}(K^*)$ over K^* for every P^*. The maps $\varepsilon_P \mid R^*$ and ε_P^* induce embeddings of $Z(R^*/P^*)$ into $\tilde{K}$ and K^*. By Lemma 2 we know $\text{t.d.}_{k^*} Z(R^*/P^*) = \dim P^* = \dim P = t$, which is also the transcendence degree of $\tilde{K}$ and K^* over k^*. Thus $\tilde{K}$ and K^* are algebraic over the two images of $Z(R^*/P^*)$. We obtain an automorphism of $\tilde{K}$ which produces a commutative square

$$\begin{array}{ccc} Z(R^*/P^*) & \longrightarrow & \tilde{K} \\ \downarrow & & \downarrow \\ K^* & \subset & \tilde{K}. \end{array}$$

By (IV) there exists then an automorphism σ of $M_{n_P}(\tilde{K})$ such that $\sigma\varepsilon_P \mid R^* = \varepsilon_P^*$. The homomorphism $\sigma\varepsilon_P: R \to M_{n_P}(\tilde{K})$ is an AI-representation for P with $\sigma\varepsilon_P(s) = \varepsilon_P^*(s) \in M_{n_P}(K^*)$ for all $s \in S$. ∎

8. Generalization of the Main Lemma

Here we shall remove the assumption that we are dealing with an algebra over a field. We start with a technical definition:

DEFINITION. In an arbitrary ring R, we call a set $\mathscr{P}$ of prime ideals *definable* ("over small fields") if, given any finite subset S of R, there exists a field F, a natural number N, and absolutely irreducible representations $\varepsilon_P: R \to M_{n_P}(\tilde{F})$ for all $P \in \mathscr{P}$, such that

(1) $\varepsilon_P(s) \in M_{n_P}(F)$ holds for all $s \in S$, and

(2) $g(\eta) \neq 0$ holds for all $\eta \in \tilde{\mathbb{P}}$ of prime degree $> N$ over $\mathbb{P}$, and all nonzero polynomials g over F of degree $\leqslant \sup\{n_P : P \in \mathscr{P}\}$.

Remarks. Condition (2) is only satisfiable if $\sup\{n_P : P \in \mathscr{P}\} < \infty$. Therefore definability of $\mathscr{P}$ implies the existence of a bound for the PI-degrees of the $P \in \mathscr{P}$. A subset of a definable set is, trivially, definable. For any ideal I of R, the set $\bar{\mathscr{P}} = \{\bar{P} : I \subset P \in \mathscr{P}\}$ of prime ideals of $\bar{R} = R/I$ is definable, via the induced $\bar{\varepsilon}_P$.

PROPOSITION 8. *In an affine noetherian PI-ring, every clique is definable.*

Proof. Let $C = Z(R)$, and note that $\Pi = C \cap P$ is independent of P, for all P is a clique $\mathscr{P}$. There is a natual map $v: R \to R_\Pi \to \bar{R} = R_\Pi/\Pi R_\Pi$, and $\bar{R}$ is an algebra over the field $k = C_\Pi/\Pi C_\Pi$; $\bar{R}$ is clearly noetherian with polynomial identity, and affine over k. By Lemma 5, $\bar{\mathscr{P}}_\Pi = \{\bar{P}_\Pi : P \in \mathscr{P}\}$ is one clique in $\bar{R}$. Therefore Lemma 6 applies to $\bar{R}$ and $\bar{\mathscr{P}}_\Pi$ and produces, for any finite subset S of R, AI-representations $\varepsilon_P: \bar{R} \to M_{n_P}(\tilde{K})$ with $\varepsilon_P v(s) \in M_{n_P}(K^{(N)})$, for all $s \in S$ and $P \in \mathscr{P}$ and all sufficiently large N.

One easily checks that $\varepsilon_P v$ is an AI-representation for P. The second condition for definability holds according to Lemma 1, if we choose N large enough to satisfy Lemma 6, as well as (N4) $N \geqslant PI$-degree of R, since then $N \geqslant \sup\{n_P : P \in \mathscr{P}\}$. ∎

9. THE MAIN THEOREM

PROPOSITION 9. *In a noetherian PI-ring, the intersection condition holds for every finite union of definable sets.*

Proof. Aiming for a contradiction, we assume that the proposition is false. By noetherian induction, we pick a minimal counterexample. This is a noetherian PI-ring R, such that in every proper factor ring the (strong version of the) intersection condition holds, while in R itself there is a finite union $\mathscr{P} = \bigcup_{i=1}^n \mathscr{P}_i$ of definable sets $\mathscr{P}_i$, and a one-sided (say right-) ideal I with the property that $I \cap \mathscr{C}(P) \neq \varnothing$ for all $P \in \mathscr{P}$ but $I \cap \mathscr{C}(\mathscr{P}) = \varnothing$.

Claim 1. $\bigcap \mathscr{P} = 0$.

Suppose not; then $\bar{R} = R/\bigcap \mathscr{P}$ is a proper factor ring. The $\bar{\mathscr{P}}_i =$

$\{\bar{P}: P\in\mathscr{P}_i\}$ are still definable, and $\bar{I}\cap\mathscr{C}(\bar{P})\neq\emptyset$ holds for all $\bar{P}\in\bar{\mathscr{P}}$. Therefore $\bar{c}\in\bar{I}\cap\mathscr{C}(\bar{\mathscr{P}})$ exists. We conclude $c\in I\cap\mathscr{C}(\mathscr{P})$; a contradiction.

Claim 2. I contains a nonzero central element α.

Let PI-degree$(R)=d$. Since R embeds into $\prod_{P\in\mathscr{P}} R/P$, by Claim 1, we can pick $P_0\in\mathscr{P}$ with PI-degree $(P_0)=d$. The Formanek polynomial for $d\times d$-matrices, $f_d(x_0,..., x_d)$, is homogeneous of degree d^2, with integer coefficients, central in R (since R is semiprime), and not identically zero in R/P_0.

We are given an element $c_0\in I\cap\mathscr{C}(P_0)$. As the quotient ring of $\bar{R}=R/P_0$ is obtainable by inverting the nonzero central elements, we have $\bar{a}\in R/P_0$ and $0\neq\beta\in Z(R/P_0)$ with $\bar{c}_0^{-1}=\bar{a}\beta^{-1}$, or $\bar{c}_0\bar{a}=\beta$. Thus $f_d(\overline{c_0ar_0},..., \overline{c_0ar_d})=\beta^{d^2}f_d(\bar{r}_0,..., \bar{r}_d)\neq 0$ for suitable $r_0,..., r_d\in R$. We obtain the non-zero central element $\alpha=f_d(c_0ar_0,..., c_0ar_d)\in I$.

(This argument is akin to the proof of Rowen's Theorem [16].)

Next, we partition $\mathscr{P}=\mathscr{P}'\cup\mathscr{P}''$, where $\mathscr{P}'=\{P\in\mathscr{P}: \alpha\in P\}$ and $\mathscr{P}''=\{P\in\mathscr{P}: \alpha\notin P\}$. In the proper factor ring $\bar{R}=R/\alpha R$, the set $\bar{\mathscr{P}}=\{\bar{P}: P\in\mathscr{P}'\}$ is the finite union of the definable sets $\bar{\mathscr{P}}_i$. We obtain an element $\bar{b}\in\bar{I}\cap\mathscr{C}(\bar{\mathscr{P}})$ in $\bar{R}$, and hence an element $b\in I\cap\mathscr{C}(\mathscr{P}')$ in R.

The definability of the $\mathscr{P}_i$ means that for the finite subset $\{\alpha, b\}$ of R there exists fields F_i (with prime fields $\mathbb{P}_i$), natural numbers N_i, and AI-representations ε_P: $R\to M_{n_P}(\tilde{F}_i)$ for all $P\in\mathscr{P}_i$, such that $\varepsilon_P(\alpha)$ and $\varepsilon_P(b)$ lie in $M_{n_P}(F_i)$, and that $g(\eta)\neq 0$ holds for all $\eta\in\tilde{\mathbb{P}}_i$ of prime degree $>N_i$ over $\mathbb{P}_i$ and all $0\neq g\in F_i[x]$ of degree $\leqslant\max\{n_P: P\in\mathscr{P}_i\}$. We fix such data for each $i=1,..., n$, and also a prime number $q>\max\{N,..., N_n\}$.

Claim 3. There is a monic polynomial of degree q over $\mathbb{Z}$, whose natural images over the $\mathbb{P}_i$ are all irreducible.

If all F_i have characteristic zero, we pick an arbitrary irreducible monic polynomial $f(x)$ of degree q over $\mathbb{Z}$; for instance, $f(x)=x^q+2$. By the Gauss Lemma, $f(x)$ is irreducible over all the $\mathbb{P}_i=\mathbb{Q}$.

If some F_i have positive characteristics p_i, we pick for each such p_i an irreducible monic polynomial $f_{p_i}(x)$ of degree q over $\mathbb{Z}/p_i\mathbb{Z}$; for instance the minimal polynomial of a multiplicative generator of $GF(p_i^q)$. We solve the systems of simultanuous congruences (modulo p_i) in $\mathbb{Z}$, for the coefficients of x^k in $f_{p_i}(x)$, for each $k=1,..., q-1$. From these solutions we construct a monic polynomial $f(x)$ of degree q over $\mathbb{Z}$, which maps onto all the $f_{p_i}(x)$. Clearly $f(x)$ is irreducible over $\mathbb{Z}$ and hence over $\mathbb{Q}$.

Let $\eta_1,..., \eta_q$ be the roots of $f(x)$ in $\tilde{\mathbb{Q}}$. They generate a subring $\mathbb{Z}[\eta_1,..., \eta_q]$ which is integral over $\mathbb{Z}$. By [1, V.2.1, Corollary 4] the natural maps $\mathbb{Z}\to\mathbb{P}_i$ extend to ring homomorphisms v_i: $\mathbb{Z}[\eta_1,..., \eta_q]\to\tilde{\mathbb{P}}_i\subset\tilde{F}_i$. The elements $v_i(\eta_k)$ are roots of the irreducible polynomials $\bar{f}$, and hence of degree q over $\mathbb{P}_i$.

The polynomial $d_P(x)=\det(\varepsilon_P(\alpha)x+\varepsilon_P(b))$, formed in $M_{n_P}(\tilde{F}_i)$, has degree $\leqslant n_P$, and coefficients in F_i since $\varepsilon_P(\alpha)$, $\varepsilon_P(b)\in M_{n_P}(F_i)$. As $\varepsilon_P(\alpha)\neq 0$ if $P\in\mathscr{P}''$, and $\det(\varepsilon_P(b))\neq 0$ if $P\in\mathscr{P}'$ because of (III), the polynomial d_P is nonzero. We conclude that $d_P(v_i(\eta_k))\neq 0$ holds for all i, k and all $P\in\mathscr{P}$. Consequently $\varepsilon_P(\alpha)\, v_i(\eta_k)+\varepsilon_P(b)$ is invertible in $M_{n_P}(F_i)$.

For each $i=1,..., n$ and $P\in\mathscr{P}$ we consider now the ring homomorphisms $R\to R\otimes\mathbb{Z}[\eta_1,..., \eta_q]\to M_{n_P}(\tilde{F}_i)\otimes\mathbb{Z}[\eta_1,..., \eta_q]\to M_{n_P}(\tilde{F}_i)$, given by $r\to r\otimes 1$, $\varepsilon_P\otimes 1$, $u\otimes\xi\to uv_i(\xi)$. (The tensor products are taken over $\mathbb{Z}$.) Their composition is simply ε_P.

In $R\otimes\mathbb{Z}[\eta_1,..., \eta_q]$ we have the element $\prod_{k=1}^q(\alpha\otimes\eta_k+b\otimes 1)=\sum_{k=0}^q\alpha^k b^{q-k}\otimes\sigma_k(\eta_1,..., \eta_q)=c\otimes 1$, where $c=\sum_{k=0}^q\alpha^k b^{q-k}\sigma_k(\eta_1,..., \eta_q)$, since α is central in R, and since the elementary symmetric polynomials $\sigma_k(\eta_1,..., \eta_q)$ are, up to the sign, the coefficients of $f(x)$ and hence lie in $\mathbb{Z}$. We deduce that $\varepsilon_P(c)=\prod_{k=1}^q(\varepsilon_P(\alpha)\, v_i(\eta_k)+\varepsilon_P(b))$ is invertible in $M_{n_P}(\tilde{F}_i)$.

We conclude $c\in\mathscr{C}(P)$ for all $P\in\mathscr{P}$. Since $\alpha, b\in I$, the definition of c yields $c\in I$ immediately. We arrive at the contradiction $c\in I\cap\mathscr{C}(\mathscr{P})=\varnothing$. ∎

Our main result is now an immediate consequence:

THEOREM 10. *In an affine noetherian PI-ring, every finite union of cliques is classical.*

Proof. According to the Propositions 8 and 9, the intersection condition holds for every finite union of cliques. Then, by [6, 8.35], such a union is classical, provided it is incomparable. In particular, single cliques are classical, since they are automatically incomparable [6, 7.26].

The comparable case is handled as in [9]: If there is $Q\in\mathscr{P}_j$, $P\in\mathscr{P}_k$ with $j\neq k$ and $Q\subsetneqq P$, then for each $Q'\in\mathscr{P}_j$ there exists $P'\in\mathscr{P}_k$ with $Q'\subsetneqq P'$. So Q' is $\mathscr{C}(\mathscr{P}_k)$-closed, and since $\mathscr{C}(\mathscr{P}_k)$ is an Ore set, we conclude $\mathscr{C}(Q')\supset\mathscr{C}(\mathscr{P}_k)$ hence $\mathscr{C}(\mathscr{P}_j)\supset\mathscr{C}(\mathscr{P}_k)$. Therefore the non-maximal cliques can be dropped without affecting $\mathscr{C}(\mathscr{P})$, and the resulting union is incomparable. ∎

10. CONSEQUENCES

If one is dealing with an algebra over an algebraically closed field, then the above proof can be somewhat shortened, and the result strengthened.

COROLLARY 11. *In an affine noetherian PI-algebra over an algebraically closed field k, the localization at a finite union of cliques has stable range one.*

Sketch of Proof. We modify the proof of Proposition 9, following the demonstration of Lemma 4.4 in [18]. As there, one obtains a regular element $a \in I$ and a nonzero ideal J such that $a \in \mathscr{C}(P)$ holds for all $P \not\supset J$. If $I = \sum_{i=1}^{m} a_i R$, then one may choose $a = a_1 + \sum_{i=2}^{m} a_i \lambda_i$ with $\lambda_i \in k$. By induction there exists $b \in I$, of the same form, with $b \in \mathscr{C}(Q)$ for all $Q \in \mathscr{P}$ with $Q \supset J$. Working with the finite subset $\{a, b\}$ as before, we obtain immediately an element $\eta \in \tilde{\mathbb{P}} \subset k$ such that $a\eta + b \in \mathscr{C}(\mathscr{P}) \cap I$. Thus the symmetrization procedure from Claim 3 onwards is unnessecary, and the element a need not be central. Clearly the element $(a\eta + b)/(1 + \eta)$ is again of the same form as a, and hence the stable range is one. ∎

In [9] we called the localization of a ring at the maximal Ore subset of the set $\mathscr{C}(0)$ of regular elements the *total quotient ring*. In generalization of Theorem 4 and Corollary 7 of that paper, we obtain from the present Theorem 10:

COROLLARY 12. *In an affine noetherian PI-ring, the maximal Ore set inside $\mathscr{C}(0)$ equals $\mathscr{C}(\mathscr{P})$, where $\mathscr{P}$is the union of the cliques of the regular prime ideals. The artinian radical of the total quotient ring is nonzero.*

REFERENCES

1. N. BOURBAKI, "Eléments de Mathématique, Algèbre Commutative," Hermann, Paris, 1964.
2. K. A. BROWN, Ore sets in enveloping algebras, *Compositio Math.* **53** (1984), 347–367.
3. K. A. BROWN, Ore sets in noetherian rings, preprint.
4. H. CARTAN AND S. EILENBERG, "Homological Algebra," Princeton Uni. Press, Princeton, N. J., 1956.
5. A. V. JATEGAONKAR, Injective modules and classical localization in noetherian rings, *Bull. Amer. Math. Soc.* **79** (1973), 152–157.
6. A. V. JATEGAONKAR, Localization in Noetherian rings, *Mem. Amer. Math. Soc.*, in press.
7. T. H. LENAGAN, Gelfand–Kirillov dimension and affine *PI*-rings, *Comm. Algebra* **10** (1982), 87–92.
8. M.-P. MALLIAVIN–BRAMERET, Dimension de Gelfand–Kirillov des algèbres à identités polynomiales, *C. R. Acad. Sci. Paris* **282** (1976), A679–681.
9. G. O. MICHLER AND B. J. MÜLLER, The maximal regular Ore set of a noetherian ring, *Arch. Math.* **43** (1984), 218–223.
10. B. J. MÜLLER, Localization in non-commutative noetherian rings, *Canad. J. Math.* **28** (1976), 600–610.
11. B. J. MÜLLER, Localization in fully bounded noetherian rings, *Pacific J. Math.* **67** (1976), 233–245.
12. B. J. MÜLLER, Ideal invariance and localization, *Comm. Algebra* **7** (1979), 415–441.
13. B. J. MÜLLER, Two-sided localization in noetherian *PI*-rings, *J. Algebra* **63** (1980), 359–373.
14. B. J. MÜLLER, The quotient problem for noetherian rings, *Canad. J. Math.* **33** (1981), 734–748.

15. C. PROCESI, "Rings with Polynomial Identities," Marcel Dekker, New York (1973).
16. L. ROWEN, Some results on the centre of a ring with polynomial identity, *Bull. Amer. Math. Soc.* **79** (1973), 219–223.
17. L. W. SMALL, Localization in *PI*-rings, *J. Algebra* **18** (1971), 269–270.
18. J. T. STAFFORD, The Goldie rank of a module, preprint.
19. M. H. UPHAM, Right localizable multiplicative sets and prime ideals of FBN rings, *Comm. Algebra* **9** (1981), 1093–1103.
20. R. B. WARFIELD, Abstract, private letter, June 1983.